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ON A DISTINCT INVARIANCE IN THE MULTIFRACTAL THEORY OF MOTION

BY

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Abstract. By using a distinct invariance of a multifractal equation of Schrödinger type, the homographic action of a 2×2 matrix with real elements, in the absolute space, is developed. In these conditions, through a Ricatti-type gauge, a Riemannian-type geometry is constructed. Then, through a geodesic Lagrangean associated to this Riemannian-type geometry, the Conservation Laws are obtained. This is a different approach from the usual one (Classical Mechanics, General Relativity, Quantum Mechanics etc.) in the tackling of problems of dynamics in multiparticle systems.

Keywords: Multifractal Theory of Motion; homographic transformation; Ricatti-type gauge; Riemannian geometry; absolute space.

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1. Introduction

It is well-known that, in the Multifractal Theory of Motion (Mercheş and Agop, 2016), dynamics related to any complex system (Badii, 1997), can be described by a multifractal Schrödinger-type equation:

$$\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \partial^l \partial_l \Psi + i\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial_t \Psi = 0 \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial_t}, \qquad \partial_l = \frac{\partial}{\partial X^l}, \qquad \partial_l \partial^l = \frac{\partial}{\partial X^l}\left(\frac{\partial}{\partial X^l}\right) \tag{2}$$

In the above relation, ψ is the multifractal state function, X^l with $l = 1,2,3$ are the multifractal spatial coordinates,t is a non-multifractal temporal coordinate having the affine parameter role on the movement curves, dt is the scale resolution, λ is a coefficient associated to the multifractal-non-multifractal transition, $f(\alpha)$ is the singularity spectrum of order α and α is the singularity index through which the fractal dimension D_F is specified (for D_F it is possible to use use any definitions – Kolmogorov fractal dimension, Hausdorff–Besikovich fractal dimension, etc. (Mandelbrot, 1982); it is regularly found that D_F< 2 for correlative processes and D_F> 2 for non-correlative processes).

From such a perspective, through $f(\alpha)$ it is possible to identify not only the dynamics that are characterized by a certain fractal dimension (*i.e.* the case of monofractal dynamics) but also the dynamics for which the fractal dimension is situated in an interval of values (*i.e.* the case of multifractal dynamics). More than that, for the same $f(\alpha)$, it is possible to identify classes of universality in the dynamics laws, even when regular or strange attractors have various aspects (Cristescu, 2008).

The Eq. (1) admits a special invariance, given by the transformations (Mazilu and Agop, 2014):

$$X' = \frac{X}{\gamma t + \delta}, \qquad t' = \frac{\alpha t + \beta}{\gamma t + \delta} \tag{3a,b}$$

where α, β, γ and δ are real elements.

The transformation (3b) can be related to the homographic action of the matrix $\boldsymbol{\alpha}$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{4}$$

in a problem that often occurs when considering that action.

In the present paper, using a distinct invariance of the multifractal Schrödinger equation in the Multifractal Theory of Motion, the following problem is analyzed: finding the relation between an ensemble of matrices $\boldsymbol{\alpha}$ and a set of values t,entering the equation (3b) in such a way that the value of t' remains constant.

2. Mathematical Procedure

The problem that needs to be solved is the following: finding the relation between an ensemble of matrices α and a set of values $\boldsymbol{t}$, entering the Eq. (3b) in such a way that the value of $\boldsymbol{t}'$ remains constant. From a geometrical point of view, this is the problem of determining an ensemble of points ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ in the absolute space as defined above, and a set of values $\boldsymbol{t}$ corresponding to that ensemble. The ensemble may be a geometrical locus, a family of lines in space, etc. Using Eq. (3b) the answer to this problem is a Riccati differential equation, obtained as a direct consequence of the condition $\boldsymbol{dt}' = \mathbf{0}$:

$$dt + \omega_1 t^2 + \omega_2 t + \omega_3 = 0 \tag{5}$$

where the following notation is used

$$\omega_1 = \frac{\gamma d\alpha - \alpha d\gamma}{\alpha\gamma - \beta\delta}, \qquad \omega_2 = \frac{\delta d\alpha - \alpha d\delta + \gamma d\beta - \beta d\gamma}{\alpha\gamma - \beta\delta}, \qquad \omega_3 = \frac{\delta d\beta - \beta d\delta}{\alpha\gamma - \beta\delta} \tag{6}$$

It is then easy to see that the metric

$$(ds)^2 = \frac{(\delta d\alpha - \alpha d\delta + \gamma d\beta - \beta d\gamma)^2}{4(\alpha\delta - \beta\gamma)^2} - \frac{d\alpha d\delta - d\beta d\gamma}{\alpha\delta - \beta\gamma} \tag{7}$$

is in relation with the discriminant of quadratic polynomial from Eq. (5), namely:

$$(ds)^2 = \frac{1}{4}(\omega_2^2 - 4\omega_1\omega_3) \tag{8}$$

The three differential forms from Eq. (6) constitute what is usually termed as a *coframe of reference* (Mazilu and Agop, 2014) in any point of the absolute space. This coframe allows to transfer geometrical properties of the absolute space into algebraic properties related to differential Eq. (5).

The simplest of these properties is the motion of a point in absolute space along the geodesics of the metric. In this case, due to particular algebraic aspect of the problem, the differential forms $\omega_1, \omega_2, \omega_3$ are exact differentials of the same parameter, which can be taken as the arc length s of a family of geodesics. The Eq. (5) then changes into an ordinary differential equation of Riccati type:

$$\frac{dt}{ds} = a_1 t^2 + 2a_2 t + a_3 \tag{9}$$

Here, the parameters a_1, a_2, a_3 are constants characterizing a given geodesic from the family. The parameter t can be found as a function of the arc length of geodesics simply by solving this equation.

It can be noticed that, if there are two known solutions of the equation, then the general solution is obtained by a single integration (Mazilu and Agop, 2014). There are indeed two known solutions of (9): the roots of the quadratic polynomial from the right hand side of the equation that are denoted by t_1, t_2. Then the ratio

$$w(s) = \frac{t(s) - t_1}{t(s) - t_2} \tag{10}$$

satisfies the first order differential equation

$$w'(s) = a_1(t_1 - t_2)w(s) \tag{11}$$

which can be integrated immediately. Thus the general solution of the Eq. (9) can be written in the form

$$t(s) = \frac{\left[t_1 - t_2 e^{a_1(t_1-t_2)s}\right]t(s_0) + t_1 t_2\left(e^{a_1(t_1-t_2)s} - 1\right)}{-\left(e^{a_1(t_1-t_2)s} - 1\right) - \left[t_2 - t_2 e^{a_1(t_1-t_2)s}\right]} \tag{12}$$

with $\left(t(s_0)\right)$ the initial condition along the particular geodesic characterized by constants a_1, a_2, a_3 . It is worth noticing that Eq. (12) represents the homographic action upon the initial value $\left(t(s_0)\right)$, as exerted by the following matrix:

$$\begin{pmatrix} t_1 & -t_1 t_2 \\ 1 & -t_2 \end{pmatrix} - e^{a_1(t_1-t_2)s}\begin{pmatrix} t_2 & -t_1 t_2 \\ 1 & -t_1 \end{pmatrix} \tag{13}$$

This matrix is a linear combination (a linear bundle) between two matrices representing points located on the absolute of space, *i.e.* it represents a point on a straight line joining two points of the absolute. With s varying in Eq. (13), one covers a segment of this straight line, possibly the whole segment inside the absolute. This line directly offers the geometric figure used before in the construction of the absolute metric. It also shows that it is worth studying the analytical meaning of a general straight line of this absolute space.

In order to do this, let it be noticed that the homographic action (3b) of a 2×2 matrix has, regardless of the nature of its entries, two fixed points, *i.e.*

two points each of which is transformed into itself by the homographic action of the matrix. These points are denoted by ξ and η. On the other hand, it is known that the cross ratio of the quadruple formed by any two points correlated through a homography and the two fixed points of the homography is a constant uniquely characterizing the homography – the so-called characteristic cross ratio. This fact gives an opportunity to represent the homographic action by a matrix well suited for a geometrical interpretation in the absolute space. Indeed, k represents the characteristic cross ratio of the homographic action of a 2×2 matrix. Then, according to the above-mentioned property

$$k = \frac{t' - \xi}{t' - \eta} : \frac{t - \xi}{t - \eta} \tag{14}$$

for any t and t' from the range of action, which are correlated by the homography. When expressing this function as in Eq. (3b), the following representation of the matrix is found:

$$\frac{\alpha}{\xi - \eta k} = \frac{\beta}{(k-1)\xi\eta} = \frac{\gamma}{1-k} = \frac{\delta}{\xi k - \eta} \tag{15}$$

This writing makes it more obvious that in homographic action only three parameters are essential. However, it expresses “more”, namely that these parameters can be taken as the two fixed points of the homography and its characteristic cross ratio. It also points out that a matrix can be written as a linear combination between two matrices of null determinant, representing points located on the absolute of this space. Indeed, (15) allows the matrix to be written in the form:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = a \cdot \left[\begin{pmatrix} \xi & -\xi\eta \\ 1 & -\eta \end{pmatrix} - k \begin{pmatrix} \eta & -\xi\eta \\ 1 & -\xi \end{pmatrix} \right] \tag{16}$$

where "a" is an arbitrary factor. This is actually the form from Eq. (13), with a particular form of k. For the variable k, it is obtained a straight line joining two points on absolute. This representation has a direct geometrical interpretation: a matrix is a point on a straight line; its non-homogeneous coordinate on that line is the characteristic cross ratio of the corresponding homography. It can then be shown that, in cases where they are incident, the straight lines (13) and (16) are orthogonal in the dot product given by Mazilu and Agop (2014).

3. The Conservation Laws

It is here the place where an interesting occurrence bearing with the three-dimensionality of the tangent space in any point takes place, namely the occurrence of the conservation laws. These come up in a pure geometrical fashion, due to the particularity of the algebraic structure. In order to reveal them, the differential forms from Eq. (6) in the representation (15) for points in this space can be written. Therefore

$$\begin{aligned}\omega_1 &= \frac{1}{\xi-\eta}\frac{dk}{k} - \frac{k-1}{k}\frac{d\xi - kd\eta}{(\xi-\eta)^2}\\ \omega_2 &= -\frac{\xi+\eta}{\xi-\eta}\frac{dk}{k} + 2\frac{k-1}{k}\frac{\eta d\xi - k\xi d\eta}{(\xi-\eta)^2}\\ \omega_3 &= \frac{\xi\eta}{\xi-\eta}\frac{dk}{k} - \frac{k-1}{k}\frac{\eta^2 d\xi - k\xi^2 d\eta}{(\xi-\eta)^2}\end{aligned} \tag{17}$$

The metric from Eq. (8) then becomes

$$(ds)^2 = \left(\frac{dk}{2k}\right)^2 + \frac{(k-1)^2}{k}\frac{d\xi d\eta}{(\xi-\eta)^2} \tag{18}$$

When considering it as a Lagrangian – the so-called geodesic Lagrangian – this metric reveals the following momentum vector:

$$p_k = \frac{dk}{2k^2}, \qquad p_\xi = \frac{(k-1)^2}{k}\frac{d\eta}{(\xi-\eta)^2}, \qquad p_\eta = \frac{(k-1)^2}{k}\frac{d\xi}{(\xi-\eta)^2} \tag{19}$$

According to known theory (Mazilu and Agop, 2014), the projection of this vector along a Killing vector of the metric generating the geodesic Lagrangian, is a conserved quantity for the motion along geodesics. This must be the situation with the differential forms (17), if the Riccati Eq. (9) is to accept any physical meaning at all: each one of those differential forms must be the projection of momentum vector (19) along a Killing vectors of the metric (18). The main idea is to find those vectors – forming actually a reference frame – and then check if they are Killing vectors or not. Writing them in the form

$$B_i = b_i^1\frac{\partial}{\partial x^1} + b_i^2\frac{\partial}{\partial x^2} + b_i^3\frac{\partial}{\partial x^3}, \qquad i = 1,2,3 \tag{20}$$

where the coordinates are labeled as$x^1 \equiv k, x^2 \equiv \xi, x^3 \equiv \eta$, then

$$\omega_i = p_k b_i^1 + p_\xi b_i^2 + p_\eta b_i^3, \qquad i = 1,2,3 \tag{21}$$

Here an explanation is in order: according to the dot product induced by the metric (8) there must be

$$\langle\omega|B\rangle = \omega_2 B_2 - 2(\omega_1 B_3 + \omega_3 B_1) \tag{22}$$

where $|\ldots\rangle$ means a 3×1 matrix and $\langle\ldots|$ means a 1×3 matrix of whatever objects the letters inside indicate – the "Dirac notation". In this case the letters indicate the vectors (20) and the differential forms (17) respectively. Consequently, in retrieving the tangent frame corresponding to the given coframe formula (21) needs to be taken into consideration, in the sense that, depending on the method, when passing from the frame to coframe the indices 1 and 3 are liable to switch places. Now the components $\left(b_i^{\ j}\right)$ can be found by identification of (21) with each one of the differential forms (17). The final result is

$$\begin{aligned}
B_1 &= 2\frac{1}{\xi-\eta}k\frac{\partial}{\partial k} + \frac{k}{k-1}\frac{\partial}{\partial \xi} - \frac{1}{k-1}\frac{\partial}{\partial \eta} \\
B_2 &= \frac{\xi+\eta}{\xi-\eta}k\frac{\partial}{\partial k} + \frac{k}{k-1}\xi\frac{\partial}{\partial \xi} - \frac{1}{k-1}\eta\frac{\partial}{\partial \eta} \\
B_3 &= 2\frac{\xi\eta}{\xi-\eta}k\frac{\partial}{\partial k} + \frac{k}{k-1}\xi^2\frac{\partial}{\partial \xi} - \frac{1}{k-1}\eta^2\frac{\partial}{\partial \eta}
\end{aligned} \tag{23}$$

These tangent vectors are satisfying the commutation rules

$$[B_1, B_2] = B_1;\ \ [B_2, B_3] = B_3;\ \ [B_3, B_1] = -2B_2 \tag{24}$$

which are specific to a $SL(2R)$ type algebraic structure and is taken as standard in the present work. This means that no matter what commutation relations the tangent vectors related to this structure happen to satisfy, they will be transformed to a form satisfying (18).

It can be checked now that the three operators (17) are Killing vectors of the metric (18), no matter of the nature of the parameters k, ξ and η. Indeed, a Killing vector of a metric is defined by the fact that the metric tensor is constant along it. This condition can be written in the form

$$B_i(h_{lm}) + h_{lj}\frac{\partial b_i^j}{\partial x^m} + \frac{\partial b_i^j}{\partial x^l}h_{jm} = 0, \qquad I, J = 1,2,3;\ \ (ds)^2 \equiv h_{ij}\,dx^i dx^j \tag{25}$$

Here is used the summation convention on dummy indices. As the metric tensor from Eq. (18) does not have but two non-zero independent components, these conditions can be easily verified, by a little tedious but otherwise direct calculation.

4. Conclusions

The main conclusions of the paper are:

i) Starting from a distinct invariance of the multifractal Schrödinger-type equation, the homographic action in one dimension is analyzed.

ii) The applied mathematical procedure is reducible to a Ricatti-type gauge, *i.e.* to a Ricatti-type differential equation.

iii) Through a Riemannian-type geometry, developed in the absolute space, the conservation laws associated to a geodesic Lagrangean are obtained.

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ASUPRA UNEI INVARIANŢE SPECIALE ÎN TEORIA MULTIFRACTALĂ A MIŞCĂRII

(Rezumat)

Utilizând o invarianță specială a ecuației multifractale de tip Schrödinger, se dezvoltă, in spațiul absolut, o acțiune omografică a matricilor 2×2 cu elemente reale. În aceste condiții, printr-o invarianță de tip Ricatti, se construiește o geometrie de tip Riemannian. Apoi, prin intermediul unui Lagrangean geodezic asociat acestei geometrii, se obțin Legile de Conservare ale Mișcării. O asemenea procedură este total diferită de cele clasice (Mecanică Clasică, Relativitatea Generală, Mecanica Cuantică etc.), în abordarea problemelor actuale din dinamica sistemelor multiparticule.